Brachistochrone problem inside the Earth revisited

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The brachistochrone, or the quickest descent path under the uniform gravity is well-known, the solution of which is a cycloid. We consider here the brachistochrone between the two points on the Earth surface which are connected by a tunnel. For this purpose the modified Fermat's priciple is employed for the spherically symmetric velocity field.

Historically the brachistochrone problem, or the quickest decent under the uniform gravity was posed first by Johann Bernoulli in June 1696, and the problem was solved by famous mathematicians of the time such as Leibnitz, Jacob Bernoulli, Newton and de L'Hopital in May 1697 [1].

The answer is a cycloid connecting the starting point A and the destination B, and the time required to pass from point A to B is given by $\sqrt{2\pi l/g}$, where l is the distance between A and B. The cycloid is a curve produced by a point P on a circumference of a circle when it rotates on the horizontal plane without slippage, and its parametric representation with circle radius a is given by

$$x = a(\varphi - \sin\varphi), \qquad y = a(1-\cos\varphi). \tag{1}$$

where φ is the rotation angle of the circle. If the distance between A and B is l, then the radius is is given by $a = l/(2\pi)$.

This answer is easily achieved if one notice the so-called Fermat's principle of the minimum time which states that light chooses the fastest path when it passes from a point to another. When the space is seperated into the the media 1 and 2, and the incident angle to the interface from the media 1 to 2 is denoted as α_1, and the refraction angle in the medium 2 as α_2, then Fermat's principle says that the path should satisfy the condition

$$\frac{v_1}{\sin\alpha_1} = \frac{v_2}{\sin\alpha_2}. \tag{2}$$

This is equivalent to Snell's law of refraction

$$n_1 \sin\alpha_1 = n_2 \sin\alpha_2,$$

if one notes that the light velocicy v is given by $v = c/n$ where c is the light velocity in the vacuum and the refraction indices of the media are n_1 and n_2, respectively.

We now consider the brachistochrone problem inside the Earth [2]. The gravity force inside of the Earth is proportional to the distance r from the center of the Earth, similar to the Hooke's law for the spring: $f_r = -kr$, where $k = -mg/R$ with m the mass of the particle, $g(\approx 9.8\mathrm{m/s^2})$ the gravity acceleration constant and R the radius of the Earth. This can be proved by the direct calculation of the gravity force, or more concisely by using the Newton-Gauss theorem which appers in the standard course of electrodynamics.

The velocity v of a partricle at r from the center is given by

$$\frac{1}{2}v^2 + \frac{1}{2}\omega^2 r^2 = \text{const.}$$

from the energy conservation law for the spring, where $\omega = \sqrt{g/R}$. If the initial velocity at the entrance A of the tunnel on the Earth surface is 0, then const. $= \omega^2 R^2/2$, and

$$v = \omega\sqrt{R^2 - r^2}. \tag{3}$$

For the quickest path, we only need to consider the path on the bisecting plane passing through the points A, B and the Earth center O. Fermat's principle (2) must be modified for this case of the circularly symmetric velocity in the following way:

$$\frac{v}{r\sin\alpha} = \text{const.} \tag{4}$$

This can be obtained from the Euler-Lagrange equation of the variational calculus. Or more elementarily it is

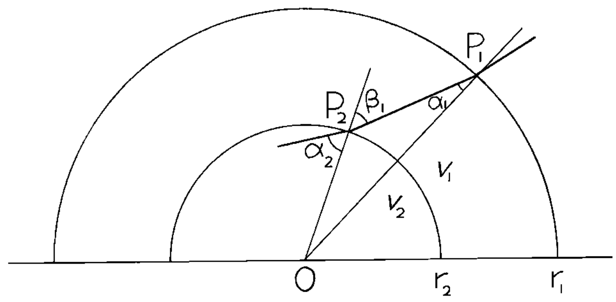

Fig. 1. Path for the spherically symmetric velocity field.

deduced from the usual refraction law by considering three media bounded by two concentric circles. In Fig. 1 the particle travels on a straight line in the medium in the middle, but the angle of refraction α_1 at P_1 and the angle of incidence β_1 at P_2 where P_1 and P_2 are on the boundary surfaces of the medium are not equal because of the different curvatures of the boundary circles. This means that the refraction angle into the medium is not directly translated into the incident angle to the next medium, in contrast to the boundaries of the parallel lines. By applying the sine theorem to the triangle made by the center of the circles O and the two points P_1 and P_2 on the trajectory line, we have

$$\frac{\overline{\mathrm{OP}_2}}{\sin\alpha_1} = \frac{\overline{\mathrm{OP}_1}}{\sin\beta_1},$$

and it is seen that the correction factor of the radius becomes necessary as in (4).

Let OA be the x axis, and let the trajectory P of the particle be expressed by $r = r(\theta)$ in the circular coordinates. Then the sine of the angle of incidence α of the trajectory is given by

$$\sin\alpha = \frac{r\mathrm{d}\theta}{\sqrt{(\mathrm{d}r)^2 + (r\mathrm{d}\theta)^2}} = \Big[\Big(\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta}\Big)^2 + 1\Big]^{-1/2}.$$

The modified Fermat's principle (4) tells that

$$\frac{1}{r}\sqrt{R^2 - r^2}\sqrt{\Big(\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta}\Big)^2 + 1} = \text{const.} = c. \tag{5}$$

From this, we obtain

$$(\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta})^2 = \frac{(c^2+1)r^2 - R^2}{R^2 - r^2}.$$

To determine the constant c, we choose the minimum of r to be R_m, then we have $c^2 + 1 = (R/R_m)^2$, and so

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \pm\frac{R}{R_m} r\sqrt{\frac{r^2 - R_m^2}{R^2 - r^2}}. \tag{6}$$

From this differential equation (6), we have

$$\theta = -\frac{R_m}{R}\int_R^r \frac{\mathrm{d}r}{r}\sqrt{\frac{R^2 - r^2}{r^2 - R_m^2}}$$

by choosing the appropriate sign. This can be integrated by changing the variable from r to φ by

$$r^2 = \frac{1}{2}\big[(R^2 + R_m^2) + (R^2 - R_m^2)\cos\varphi\big], \tag{7}$$

and the result is

$$\theta = \tan^{-1}\big(\frac{R_m}{R}\tan\frac{\varphi}{2}\big) - \frac{R_m}{2R}\varphi. \tag{8}$$

The path of the tunnel $r = r(\theta)$ is expressed parametrically by (7) and (8).

The time t required for the particle to travel from A to the point P is given by

$$
\begin{aligned}
t &= \int_{\mathrm{A}}^{\mathrm{P}} \frac{\mathrm{d}s}{v} = \frac{1}{\omega}\int_R^r \mathrm{d}r \frac{\mathrm{d}\theta}{\mathrm{d}r}\sqrt{\frac{(r')^2 + r^2}{R^2 - r^2}} \\
&= \frac{1}{\omega}\frac{\sqrt{R^2 - R_m^2}}{R}\int_r^R \mathrm{d}r r \frac{1}{\sqrt{(R^2 - r^2)(r^2 - R_m^2)}}.
\end{aligned}
$$

This is integrated by the use of (7) to yield

$$
t = \frac{1}{2\omega}\sqrt{1 - (\frac{R_m}{R})^2}\ \varphi. \tag{9}
$$

That is, the angular velocity $\mathrm{d}\varphi/\mathrm{d}t$ is constant. The time T to take from A to B corresponds to $\varphi = 2\pi$, and so

$$
T = \frac{\pi}{\omega}\sqrt{1 - (\frac{R_m}{R})^2}. \tag{10}
$$

In the case of the straight tunnel passing through the Earth center, we can set $R_m = 0$ to obtain $T_s = \pi/\omega$.

The curve obtained in (7) and (8) appears to be complicated but has a simple geometrical interpretation. In fact this curve is called hypo-cycloid, a generalized curve from cycloid. Hypo-cycloid is the curve produced by a point P on a small circle O with radius a when it rolls inside of a larger circle O$'$ with radius R by touching the perimeter (see Fig. 2). Let the rotating angle of the smaller circle be φ, then the curve in the parametrical representation is

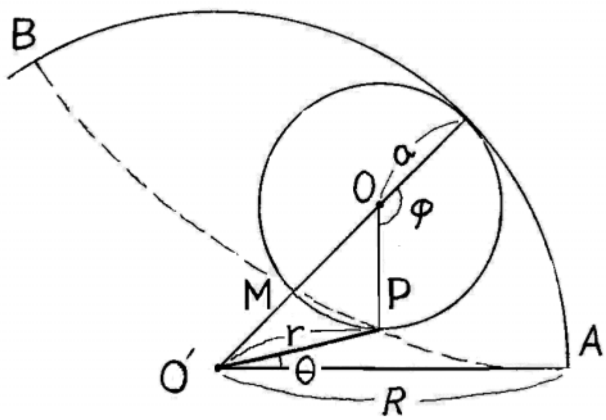

Fig. 2. Hypo-cycloid.

given by

$$
\begin{aligned}
x &= (R-a)\cos(\frac{a}{R}\varphi) + a\cos\big((1-\frac{a}{R})\varphi\big) \qquad (11)\\
y &= (R-a)\sin(\frac{a}{R}\varphi) - a\sin\big((1-\frac{a}{R})\varphi\big). \qquad (12)
\end{aligned}
$$

In Fig. 2, it will be clearly understood if we notice that $\boldsymbol{r} = \overrightarrow{\mathrm{O'P}} = \overrightarrow{\mathrm{O'O}} + \overrightarrow{\mathrm{OP}}$ and $\angle \mathrm{AO'O} = a\varphi/R$ which is obtained from the rolling condition of the smaller circle without slippage. These will be shown to be equivqlent to the set (7) and (8) in the following way.

If we take $a = (1/2)(R - R_m)$ in the above equation for a hypo-cycloid, then we have

$$
\begin{aligned}
x &= r\cos\theta = R\ \big[\cos\frac{\varphi}{2}\cos\frac{\rho\varphi}{2} + \rho\sin\frac{\varphi}{2}\sin\frac{\rho\varphi}{2}\big], \\
y &= r\sin\theta = R\ \big[\rho\sin\frac{\varphi}{2}\cos\frac{\rho\varphi}{2} - \cos\frac{\varphi}{2}\sin\frac{\rho\varphi}{2}\big],
\end{aligned}
$$

where $\rho = R_m/R$. From this, (7) will be obtained easily if we calculate $r^2 = (x^2 + y^2)$. Devision of the each side of the above equations will lead to

$$\tan\theta = \frac{x}{y} = \frac{\rho\tan(\varphi/2) - \tan(\rho\varphi/2)}{1 + \rho\tan(\varphi/2)\tan(\rho\varphi/2)} = \tan\left(\tan^{-1}(\rho\tan\frac{\varphi}{2}) - \frac{\rho\varphi}{2}\right),$$

and this is seen to be equivalent to (8). Hence, the tunnel trajectory given by (7) and (8) is a hypo-cycloid.

Geometrically, the angle $\theta + (\rho\varphi/2)$ is seen to be equal to $\angle$O′PM, if the cross point of the line $\overline{\mathrm{O'O}}$ and circle O is denoted as M. The motion of the particle follows P of the small circle O when it rotates with the constant angular velocity, $\mathrm{d}\varphi/\mathrm{d}t = 2\omega/\sqrt{1-\rho^2}$. The velocity of point P is given by $\omega\sqrt{R^2 - R_m^2}\sin(\varphi/2)$, and directs parallel along $\overrightarrow{\mathrm{PM}}$. The great circle distance l between A and B is given by $l = 2\pi a = \pi(R - R_m)$, and the angle between A and B from the Earth center, $\Theta = \angle$AOB is given by $\Theta = l/R = \pi(1-\rho)$.

If the points A and B are located at the opposite positions with respect to the Earth center, and the tunnel is straight ($\rho = 0$), then the required time T_s is given by π/ω, which is roughly 41 min. This is precisely equal to the freight time of the satelite flying just above the Earth surface.

In the limit of large R, or $(a/R) \to 0$, the corespondence $R - x \to y$, $y \to x$ made in (11) and (12) leads to the cycloid of (1). It should be noted that hypo-cycloid has also the tautochrone property under the harmonic potential similar to cycloid under the uniform gravity. We leave the proof for the reader, as it can be proved easily.

References

[1] http://www-groups.dcs.st-andrews.ac.uk/˜history/HistTopics/Brachistochrone.html

[2] E.J. Routh, *"A Treatise on Dynamics of a Particle"*, (Cambridge University Press, 1898).